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ANOTHER COMPUTATIONAL APPROACH TO A MATHEMATICAL MODEL OF TURBULENCE

S. P. Azen, R. Bellman and J. M. Richardson

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The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

Part of the research program of The RAND Corporation consists of basic supporting studies in mathematical physics. One area of this is concerned with the study of turbulence. In this field, the analysis of "model equations" has proved to be quite useful.

In a series of Memoranda, of which this is the second,* computational techniques which approximate the solution to the model equation of Burgers are investigated.

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*See Ref. 2.

SUMMARY

The nonlinearity of the equations of turbulence forces the mathematician to study approximation techniques, such as "model equations" which exhibit many of the characteristics of the realistic equations.

In this series, the authors investigate computational methods to the solution of the model equation of Burgers. In the present Memorandum the application of a difference algorithm is studied. Numerical results obtained from a FORTRAN program are given.

ANOTHER COMPUTATIONAL APPROACH TO A MATHEMATICAL MODEL OF TURBULENCE

1. INTRODUCTION

In the study of turbulence it is often quite useful to analyze "model equations" which, hopefully, exhibit many of the characteristics of the more realistic equations. One of the best known of these model equations is that of Burgers which is given by

$$
\begin{aligned}
&u_t + uu_x = \epsilon u_{xx} \\
&u(x,0) = g(x) \ .
\end{aligned}
\tag{1.1}
$$

This equation is discussed by Hopf [1], and in the first paper of this series [2] a numerical technique is presented in which this equation is converted into an infinite system of ordinary differential equations.

In this second paper, another numerical technique is investigated. Here, an approximating algorithm is used which estimates the solution of (1.1) to an accuracy of at least $O(\Delta^2)$, where Δ is the integration stepsize. Similar methods were studied in previous papers [3,4], and shown to be quite useful.

2. AN APPROXIMATING ALGORITHM

Consider the equation (1.1) over the region $0 \leq x \leq 1$, $t > 0$, and suppose that $g(x)$ is periodic, with period π. Let the approximating algorithm be given by

$$u(x,t+\Delta) = \lambda u(x-au(x,t)\Delta,t) + \frac{(1-\lambda)}{2}[u(x+b\sqrt{\Delta},t) + u(x-b\sqrt{\Delta},t)] \tag{2.1}$$

where Δ is the integration stepsize, and λ, a, b are constants which will be determined. To show that (2.1) approximates (1.1) to an error of $O(\Delta^2)$, expand both sides of (2.1) in a Taylor series to the Δ^2 term, obtaining the equation,

$$u_t = -\lambda a u u_x + (1-\lambda)\frac{b^2}{2} u_{xx} . \tag{2.2}$$

For (2.2) to approximate (1.1), the following relations must hold:

$$a = 1/\lambda \tag{2.3}$$

$$b = \left(\frac{2\epsilon}{1-\lambda}\right)^{1/2} . \tag{2.4}$$

If ϵ is fixed, then a and b are functions of the parameter λ, and (2.1) becomes

$$u(x,t+\Delta) = \frac{1}{\lambda}\, u(x-u(x,t)\,\frac{\Delta}{\lambda},\, t) + \frac{1-\lambda}{2}\left[u\left(x+\left(\frac{2\epsilon\Delta}{1-\lambda}\right)^{1/2},t\right) + u\left(x-\left(\frac{2\epsilon\Delta}{1-\lambda}\right)^{1/2},t\right)\right] . \tag{2.5}$$

Let $t = 0,\Delta,2\Delta,\ldots$, and at each stage of the calculation let $u(x,t)$ be stored by means of the finite sum

$$u(x,t) \cong \sum_{n=1}^{M} u_n(t) \sin n\pi x \tag{2.6}$$

where the coefficients $u_n(t)$ are obtained by the quadrature scheme

$$u_n(t) = 2 \int_0^1 u(x,t) \sin n\pi x \, dx \tag{2.7}$$

$$\cong \frac{2}{R} \sum_{k=1}^{R-1} u(k/R,t) \sin (n\pi k/R) \quad . \tag{2.8}$$

Hence, the values $u(k/R,t)$, $k = 1,2,\ldots,R-1$ store $u(x,t)$ at time t, and by way of (2.5), $u(x,t+\Delta)$ can be obtained.

3. NUMERICAL EXAMPLES

To obtain some numerical results, a FORTRAN program was written for the IBM-7090. The following results were obtained:

a) $u_t + uu_x = \epsilon u_{xx}$ where

$u(x,0) = -\sin \pi x, \; 0 \le x \le 1$

$\epsilon = .01$

$\lambda = .5$

$\Delta = .05$

In this first example the parameters M and R were varied. As can be seen in the following table, one obtains essentially the same results for M = R = 10, as compared to M = R = 15, in less than half the time.

x	t	u(x,t)	
		M=R=10	M=R=15
.5	1.00	-.355	-.354
.1	2.00	-.319	-.326
.6	3.00	-.116	-.116
.9	4.00	-.022	-.022
.2	5.00	-.132	-.133
Time		40 sec	1 min

Note that in this example and in the following examples the largest differences occur for small values of x.

b) $u_t + uu_x = \epsilon u_{xx}$ where

$u(x,0) = -\sin \pi x, \quad 0 \le x \le 1$

$\epsilon = .01$

$\Delta = .05$

$M = R = 10$

In the second example the parameter λ was varied. As shown in Fig. 1, varying λ displays large differences in the values of $u(x,t)$ for small x and for small t. Note, however, that the maximum of each curve occurs at about the same time. For small t and large x, the variation in $u(x,t)$ is much smaller (see Fig. 2); and for large values of t, the variation is quite small (see the table below).

x	t	$\lambda = 2$	$\lambda = 8$	difference
.1	3	-.184	-.223	.039
.6	3	-.115	-.115	.000
.1	4	-.131	-.157	.026
.6	4	-.089	-.089	.000
.1	5	-.098	-.115	.017
.6	5	-.073	-.073	.000

c) $u_t + uu_x = \epsilon u_{xx}$ where

$u(x,0) = -\sin \pi x, \quad 0 \le x \le 1$

$\epsilon = .01$

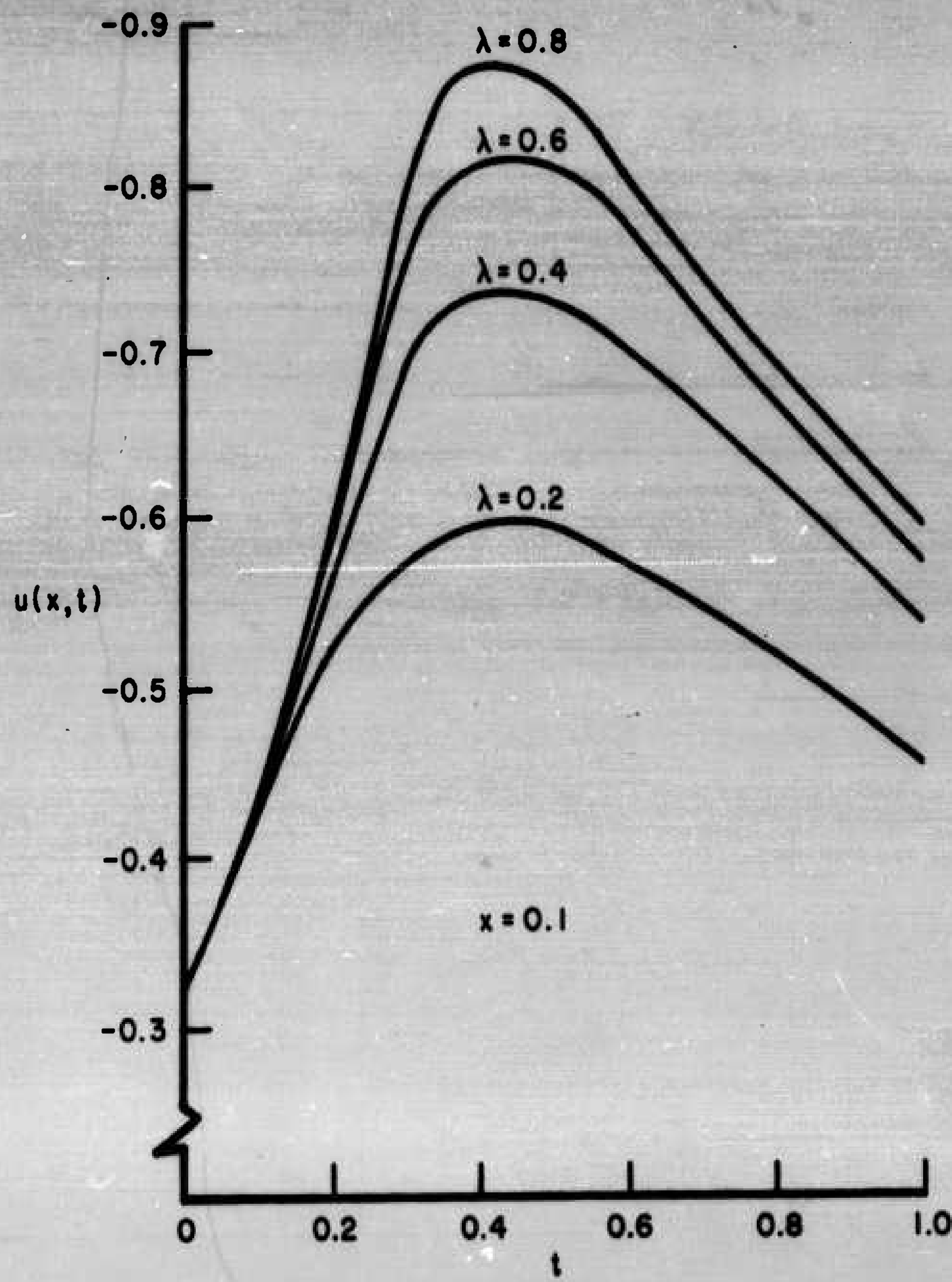

Fig. 1

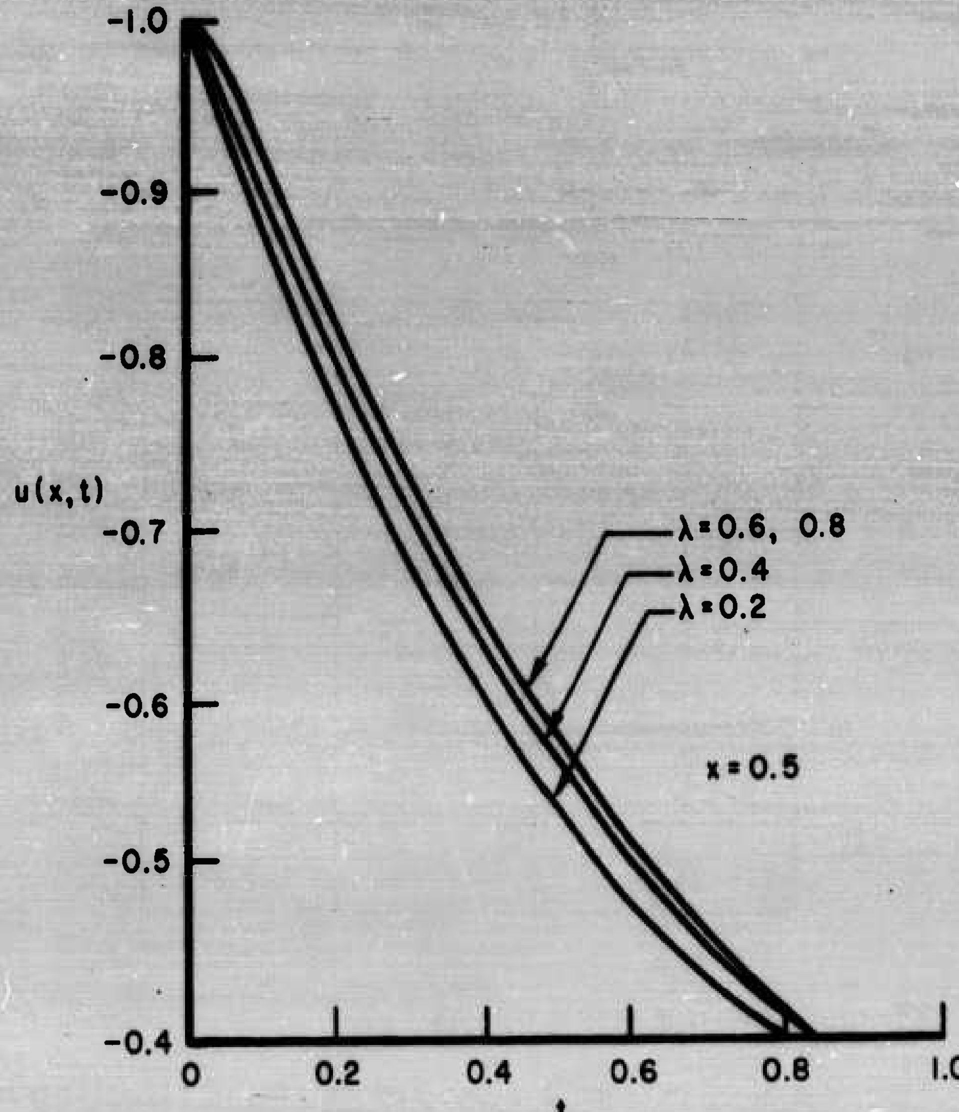

Fig. 2

$\lambda = .5$

$M = R = 10$

The parameter which has the greatest effect on the results is the integration stepsize Δ. In the following table the differences can be seen.

x	t	u(x,t)	
		$\Delta = .05$	$\Delta = .01$
0.5	1.0	-.355	-.381
0.1	2.0	-.319	-.375
0.6	3.0	-.116	-.120
0.9	4.0	-.022	-.023
0.2	4.8	-.139	-.147
Time		40 sec	180 sec

4. HIGHER-ORDER APPROXIMATION

The general form for the approximating algorithm is given by

$$u(x, t+\Delta) = \lambda u(x-au(x,t)\Delta,t) + \sum_{i=1}^{N} a_i \left[u(x+b_i\sqrt{\Delta},t) + u(x-b_i\sqrt{\Delta},t)\right] \tag{4.1}$$

in which the λ, R, a_i, b_i, and a are inputs. It was shown in [5] that higher-ordered approximations, as in (4.1), can give more-accurate results, provided the polynomial approximation (2.6) is sufficiently accurate.

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